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Structure Redesign for Improved Dynamic Response

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The concepts of multivariable control design are used to redesign structures represented by lumped parameter models. When the mass is held constant, the changing of passive structural members is mathematically equivalent to an output feedback decentralized control problem. When the mass is also allowed to change, a generalization of the output feedback formulation is required. The design objective is to minimize the changes in the structure required to simultaneously satisfy inequality constraints on both the root-mean-square and absolute value of the dynamic response of each output. An alternative objective treated is to minimize the root-mean-square of the dynamic response subject to inequality constraints on the changes allowed in each structural member. Examples illustrate both procedures.

I. Introduction

IMPROVED dynamic response of structures, or of elastic mechanical systems in general, is a frequent goal in modern engineering. Dynamic flexible motions of structures due to transient and vibratory input forces can reduce the performance of attached sensitive equipment or, in severe cases, result in failure of the attached equipment or the structure itself, due to excessive strain or fatigue. The dynamic response of a flexible structural/mechanical system may be improved in two ways: by redesigning the structure or by using active feedback control. Even if active controls are to be used, some structures are easier to control (require less control effort) than others, and to take advantage of this fact it is desirable to redesign the structure to improve the dynamic response *before* finalizing an active control design.

Classical single-degree-of-freedom vibration theory served for many decades as a guide to structural dynamists in overcoming dynamic response problems, and it remains an intuitive base for initial design. More recently, the availability of high-speed computing equipment has made on-line, person-in-the-loop, iterative design procedures practical for improving the design of relatively large elastic systems. These are readily available in commercial structural dynamics modeling software. In addition, researchers have begun to apply automated nonlinear optimization to design structures to minimize specific objective functions, usually mass or damping energy, subject to constraints on parameters such as natural frequency or structural component dimensions.[1-10] Others, including this paper, attack the problem by considering the structural mass, stiffness, and damping properties as control variables in an output feedback control problem. In the approach described herein, the structural parameter matrices are decomposed into physical connectivity form, and the resulting three matrix set is treated as an output feedback control design problem in which local mass, stiffness, and damper magnitudes are "control gains." Algorithms to solve for the gains, i.e., the physical parameters, which minimize two alternative objective functions, are developed and applied to a small lumped-mass system. Analytical techniques borrowed from the control literature[11-16] on the analysis of robustness properties of linear systems are used to accomplish the design. We will determine whether feedback control is even necessary or whether the performance can be adequately improved by structure redesign.

The paper is organized as follows. Section II casts the structure redesign problem as an output feedback problem. Section III presents the necessary robustness properties of linear systems. Section IV states two versions of the mathematical problem and presents design algorithms. Section V provides some examples, and Sec. VI discusses practical applications. Some conclusions are offered in Sec. VII.

II. Output Feedback Nature of Structure Redesign

Consider the familiar dynamic system

$$M\ddot{q} + (D + G)\dot{q} + Kq = w$$
$$y = Pq + R\dot{q} \tag{1}$$

where $y \in \mathbb{R}^{n_y}$, $q \in \mathbb{R}^{n_q}$, $M = M^* > 0$, $K = K^* \geq 0$, $D = D^* \geq 0$, $G = -G^*$. The performance requirements are to limit the response $y_i(t)$ to a specified amplitude ϵ_i. Mathematically, the problem is to keep the L_∞ norm below a certain bound

$$\|y_i(\cdot)\|_\infty^2 \triangleq \sup_t y_i^2(t) \leq \epsilon_i^2, \qquad i = 1, \ldots, n_y \tag{2a}$$

for any disturbance $w(t)$ with a bounded L_2 norm,

$$\|w(t)\|_2^2 \triangleq \int_0^\infty w^*(t)w(t)\,\mathrm{d}t \leq \beta^2 \tag{2b}$$

Assume that Eq. (1) represents the initial structure design, but suppose the dynamic response $y(t)$ to a specified disturbance $w(t)$ is not acceptable. That is, suppose Eq. (2b) is satisfied and Eq. (2a) is not satisfied. We seek to use active control only if structure redesign cannot satisfy the requirements of Eqs. (1) and (2) or if the structure redesign alternative is physically unacceptable. The best combination of structure

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redesign and active control is obtained in this way, but the subject of this paper is restricted to structure redesign. By structure redesign we mean the changing of the mass matrix by ΔM, the damping matrix by ΔD, and the stiffness matrix by ΔK to yield the new system

$$(M+\Delta M)\ddot{q} + (D+\Delta D+G)\dot{q} + (K+\Delta K)q = w \qquad (3)$$

satisfying Eq. (2).

Some structure design problems have optimized the structure with a constraint on the flexibility (usually a constraint on the frequency of the first mode of vibration), and some have used linear programming solutions to minimize quadratic objective functions.[1-10] Our objectives are much more physically motivated. It is quite natural to require the dynamic response to stay within specified bounds of Eq. (2a) for a given class of disturbances in Eq. (2b). For example, maximal strength limitations on real structures can be related to displacement amplitude, and pointing and focusing errors in an antenna relate directly to maximum allowable displacements. These kinds of performance requirements apply to almost all space systems flown to date (although the techniques used in their design have been very indirect relative to these objectives).

To illustrate the procedure, consider the three-mass system in Fig. 1, where the stiffness matrix is as follows:

$$K = \begin{bmatrix} k_1+k_4+k_5 & -k_4 & -k_5 \\ -k_4 & k_2+k_4+k_6 & -k_6 \\ -k_5 & -k_6 & k_3+k_5+k_6 \end{bmatrix} \qquad (4)$$

This matrix can be written in the form

$$K = B_k G_k B_k^* \qquad (5a)$$

where

$$B_k = \begin{bmatrix} 1 & 0 & 0 & 1 & -1 & 0 \\ 0 & 1 & 0 & -1 & 0 & -1 \\ 0 & 0 & 1 & 0 & 1 & 1 \end{bmatrix}$$

$$G_k = \mathrm{diag}[k_1, k_2, k_3, k_4, k_5, k_6] \qquad (5b)$$

where B_k is the stiffness connectivity matrix. Likewise, the damper connectivity matrix is B_d where $B_d = B_k$ in this example, since the springs and dampers are collocated. The damping matrix is thus written as

$$D = \begin{bmatrix} d_1+d_4+d_5 & -d_4 & -d_5 \\ -d_4 & d_2+d_4+d_6 & -d_6 \\ -d_5 & -d_6 & d_3+d_5+d_6 \end{bmatrix} = B_d G_d B_d^*$$

$$G_d \triangleq \mathrm{diag}[d_1, \ldots, d_6] \qquad (6)$$

Figure 2 shows the connectivity matrices for several different system configurations.

The connectivity matrices B_m, B_d, and B_k can easily be derived in general from the structured matrices M, K, and D by the singular value decomposition. This is shown as follows:

$$K = k_1\bar{K}_1 + k_2\bar{K}_2 + \cdots + k_q\bar{K}_q = \sum_{i=1}^{q} k_i\bar{K}_i$$

$$k_i \in \mathcal{R}, \quad \bar{K}_i \in \mathcal{R}^{n\times n}$$

where k_i represents one spring element in our example, and $\bar{K}_i$ is a constant symmetric matrix. Factor the symmetric $\bar{K}_i$ by the singular value decomposition

$$\bar{K}_i = U_i \Sigma_i U_i^*, \qquad \Sigma_i = \mathrm{diag}[\cdots \sigma_i \cdots] > 0$$

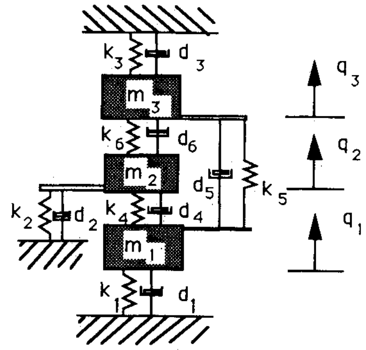

Fig. 1 Three-mass damped elastic system.

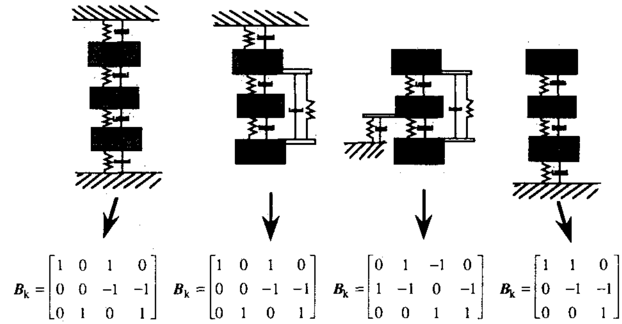

Fig. 2 Some alternative three-mass connection geometries.

where U_i is a column unitary matrix associated with the nonzero singular values σ_i. Hence

$$K = \sum_i^q k_i U_i \Sigma_i U_i^* = B_k G_k B_k^*$$

where

$$B_k = \left[U_1\sqrt{\Sigma_1},\ U_2\sqrt{\Sigma_2}, \ldots\right]$$

$$G_k = \mathrm{diag}\left[k_1 I_1, k_2 I_2, \ldots, k_q I_q\right]$$

where I_i is an identity matrix of the dimension of Σ_i. The M and D can be factored in a similar way to yield

$$M = B_m G_m B_m^*$$

$$D = B_d G_d B_d^*$$

The mass matrix is diagonal in the coordinates chosen for the Fig. 1 example, hence

$$M = \begin{bmatrix} m_1 & 0 & 0 \\ 0 & m_2 & 0 \\ 0 & 0 & m_3 \end{bmatrix} = B_m G_m B_m^* \qquad (7)$$

where $B_m = I$, $G_m = M$ in this example. Thus any change in the structural parameter k_i to $k_i + \Delta k_i$ may be expressed in the form

$$K + \Delta K = K + B_k G_k B_k^*, \quad G_k = \mathrm{diag}[\cdots \Delta k_i \cdots] \qquad (8a)$$

and any change in the dampers leads to

$$D + \Delta D = D + B_d G_d B_d^*, \quad G_d = \mathrm{diag}[\cdots \Delta d_i \cdots] \qquad (8b)$$

and any change in masses leads to

$$M + \Delta M = M + B_m G_m B_m^*, \quad G_m = \mathrm{diag}[\cdots \Delta m_i \cdots] \qquad (8c)$$

This yields the equivalent representation of Eq. (3):

$$M\ddot{q} + (D+G)\dot{q} + Kq = -\begin{bmatrix} B_k & B_d & B_m \end{bmatrix} \begin{bmatrix} G_k & 0 & 0 \\ 0 & G_d & 0 \\ 0 & 0 & G_m \end{bmatrix} \begin{bmatrix} B_k^* q \\ B_d^* \dot{q} \\ B_m^* \ddot{q} \end{bmatrix} + w = Bu + w \tag{9}$$

where $u = -Gz$, and

$$B \triangleq \begin{bmatrix} B_k & B_d & B_m \end{bmatrix}, \qquad z \triangleq \begin{bmatrix} z_k \\ z_d \\ z_m \end{bmatrix} = \begin{bmatrix} B_k^* q \\ B_d^* \dot{q} \\ B_m^* \ddot{q} \end{bmatrix}$$

$$G \triangleq \begin{bmatrix} G_k & 0 & 0 \\ 0 & G_d & 0 \\ 0 & 0 & G_m \end{bmatrix}, \qquad u \triangleq \begin{bmatrix} u_k \\ u_d \\ u_m \end{bmatrix} \tag{10}$$

The control interpretation of Eq. (10) is that u is a measurement feedback control signal where $z(t)$ is the measurement vector, composed of position, rate, and acceleration measurements, and u_k is collocated with z_k, u_d is collocated with z_d, and u_m is collocated with z_m.

Equation (9) may be put into the state form

$$\begin{aligned} \dot{x} &= Ax + Bu + Dw \\ y &= Cx \\ z &= Mx \\ u &= -Gz \end{aligned} \tag{11a}$$

where

$$x \triangleq \begin{bmatrix} q \\ \dot{q} \end{bmatrix}, \qquad A \triangleq \begin{bmatrix} 0 & I \\ -M^{-1}K & -M^{-1}(D+G) \end{bmatrix}$$

$$B \triangleq \begin{bmatrix} 0 \\ M^{-1}B \end{bmatrix}, \quad D \triangleq \begin{bmatrix} 0 \\ M^{-1} \end{bmatrix}, \quad C \triangleq \begin{bmatrix} P & R \end{bmatrix} \tag{11b}$$

$$M \triangleq (I+EG)^{-1}F$$

$$F \triangleq = \begin{bmatrix} B_k^* & 0 \\ 0 & B_d^* \\ -B_m^* M^{-1}K & -B_m^* M^{-1}(D+G) \end{bmatrix}$$

$$E \triangleq \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ B_m^* M^{-1} B_k & B_m^* M^{-1} B_d & B_m^* M^{-1} B_m \end{bmatrix}$$

Because of the structure of E, F, and G, it may be noted that

$$M \triangleq (I+EG)^{-1}F = \begin{bmatrix} B_k^* & 0 \\ 0 & B_d^* \\ -(I+B_m^* M^{-1}B_m G_m)^{-1} B_m^* M^{-1}(K+B_k G_k B_k^*) & -(I+B_m^* M^{-1} B_m G_m)^{-1} B_m^* M^{-1}(D+G+B_d G_d B_d^*) \end{bmatrix} \tag{11c}$$

Define

$$A_{CL} \triangleq A - BG(I+EG)^{-1}F = \begin{bmatrix} 0 & I \\ A_{21} & A_{22} \end{bmatrix} \tag{11d}$$

where Eqs. (11a-11c), after some manipulation, lead to

$$\begin{aligned} A_{21} &= -(M + B_m G_m B_m^*)^{-1}(K + B_k G_k B_k^*) \\ A_{22} &= -(M + B_m G_m B_m^*)^{-1}(D + B_d G_d B_d^*) \end{aligned} \tag{11e}$$

But Eq. (11e) could have been obtained directly from Eq. (3). The only purpose of Eqs. (11a-11d) is to show that the structure redesign problem can be treated mathematically as a decentralized output feedback control problem. The adjective "decentralized" refers to the fact that G has special structure. In fact, G is diagonal, which is equivalent to a control problem using only local-collocated feedback. The closed-loop system of Eq. (11a) is described by

$$\dot{x} = A_{CL}x + Dw$$

$$y_{CL} \triangleq \begin{bmatrix} y \\ u \end{bmatrix} = \begin{bmatrix} C \\ -GM \end{bmatrix} x = C_{CL}x \tag{12}$$

III. Disturbance Rejection Properties of the Redesigned Structure

Before stating the mathematical problem, we shall derive some important properties of the system of Eqs. (1) and (2).

Theorem 1

If A_{CL} is asymptotically stable, then the outputs of

$$\dot{x} = A_{CL}x + Dw, \qquad y_{CL} = C_{CL}x \tag{13}$$

satisfy the L_∞ bounds

$$\|y_{CL_i}(\cdot)\|_\infty^2 \triangleq \sup_t y_{CL_i}^2(t) \le (C_{CL}XC_{CL}^*)_{ii}\beta^2 \tag{14}$$

for any disturbance satisfying

$$\int_0^\infty w^*(t)D^*(DWD^* + X_0)^{-1}Dw(t)\,\mathrm{d}t \le \beta^2 \tag{15}$$

where X satisfies

$$0 = XA_{CL}^* + A_{CL}X + DWD^* + X_0 \tag{16}$$

The following lemma is needed for the proof of Theorem 1.

Lemma 1[17]

The asymptotically stable system

$$\dot{x} = A_{CL}x + w_{CL}, \qquad y_{CL} = C_{CL}x \tag{17}$$

satisfies the L_∞ bounds

$$\sup_t y_{CL_i}^2(t) \le (C_{CL}XC_{CL}^*)_{ii}\beta^2 \tag{18}$$

for any $w_{CL}(t)$ such that

$$\int_0^\infty w_{CL}^*(t)W_{CL}^{-1}w_{CL}(t)\,\mathrm{d}t \le \beta^2 \tag{19}$$

where X satisfies

$$0 = XA^*_{CL} + A_{CL}X + W_{CL} \tag{20}$$

Proof of Theorem 1

Lemma 1 holds for any $w_{CL}(t)$ and any invertible W_{CL}. Let

$$w_{CL} \triangleq Dw, \qquad W_{CL} \triangleq DWD^* + X_0 \qquad \square$$

to get Eqs. (15) and (16) directly.

The X in Eq. (16) has a physical interpretation[15]:

$$X = \sum_{i=1}^{nx+nw} \int_0^\infty x(i,t)x^*(i,t)\,\mathrm{d}t \tag{21}$$

where $x(i,t)$ is the response of Eq. (13) to the ith excitation. The total of $n_x + n_w$ excitations is composed of initial conditions $x_k(0)$, $k = 1,\dots,n_x$, and impulses of strength w_α, $w_\alpha = w_\alpha\delta(t)$, $\alpha = 1,\dots,n_w$, and matrices W and X_0 are defined by

$$W = [\cdots w_\alpha^2 \cdots], \qquad X_0 = \mathrm{diag}[\cdots x_k^2 \cdots] \tag{22}$$

Hence, by requiring $x_k(0) = 0$ for all $k = 1,\dots,n_x$, the inverse required in Eq. (15) is guaranteed to exist.

One of our structure design goals is to satisfy

$$\|y_{CL}(\cdot)\|_\infty^2 \le \epsilon_i^2$$

It is clear from Eq. (14) that this goal is achieved by requiring

$$(C_{CL}XC^*_{CL})_{ii} \le \epsilon_i^2/\beta^2 \triangleq \sigma_i^2 \tag{23}$$

where the physical significance of the left side is

$$C_{CL}XC^*_{CL} = \sum_{j=1}^{nx+nw} \int_0^\infty y_{CL}(j,t)\,y^*_{CL}(j,t)\,\mathrm{d}t \tag{24}$$

and the rms value of the ith output is

$$\left[\sum_{j=1}^{nx+nw} \int_0^\infty y^2_{CL_i}(j,t)\,\mathrm{d}t\right]^{1/2} = \left[\left(C_{CL}XC^*_{CL}\right)_{ii}\right]^{1/2} \tag{25}$$

The significant conclusion of this discussion is that the L_∞ bounds $\|y_i(\cdot)\|_\infty^2 \le \epsilon_i^2$ on the outputs are satisfied [for all L_2 bounded inputs in Eq. (15)] if the rms values in Eq. (25) of the outputs [for a *different* set of excitations, Eq. (22)] is bounded by ϵ_i^2/β^2. Of course, one may question the conservativeness of the upper bound of Eq. (14) before attaching too much significance to these results. It has been proven,[18] however, that Eq. (14) is the tightest possible bound and that the disturbance

$$w(t) = \begin{cases} \beta\bar{\sigma}[Y_f]^{-1/2}L^*e^{A_{CL}(t_f-t)}C^*_{CL}Y_f^+Y_fe_f, & 0 \le t \le t_f \\ 0 \quad \text{if } t > t_f \end{cases} \tag{26a}$$

leads to an equality in Eq. (14), if t_f is large enough, where

$$Y_f \triangleq \int_0^{tf} C_{CL}e^{A_{CL}}LL^*e^{A^*_{CL}t}C_{CL}\,\mathrm{d}t \tag{26b}$$

and e_f is the eigenvector of Y_f associated with its maximal value, and

$$LL^* \triangleq DWD^* + X_0 \tag{26c}$$

IV. Statement of the Structure Redesign Problem

Over the range of all disturbances with a given energy level β^2, Eq. (14) provides the best possible bound.[18] Hence, our problem is well posed as follows.

Output-Input L_2 Constrained (OIL_2) Problem

Given the system of Eq. (11), find the diagonal matrix G to minimize

$$\sum_{j=1}^{nx+nw} \int_0^\infty \left[x^*(j,t)Qx(j,t) + u^*(j,t)Ru(j,t)\right]\mathrm{d}t \tag{27}$$

subject to inequality constraints on the rms values of $y_i(t)$ and $u_k(t)$:

$$\left[\sum_{j=1}^{nx+nw} \int_0^\infty y_i^2(j,t)\,\mathrm{d}t\right]^{1/2} \le \sigma_i, \qquad i = 1,\dots,n_y \tag{28}$$

$$\left[\sum_{i=1}^{nx+nw} \int_0^\infty u_k^2(i,t)\,\mathrm{d}t\right]^{1/2} \le \mu_k, \qquad k = 1,\dots,n_u \tag{29}$$

This problem can be formulated mathematically as follows:

$$\min_{\mathrm{diag}\,G} \mathrm{tr}X(Q + M^*G^*RGM) \tag{30}$$

subject to

$$0 = XA^*_{CL} + A_{CL}X + DWD^* + X_0 \tag{31}$$

$$(C_{CL}XC^*_{CL})_{ii} \le \epsilon_i^2/\beta^2 \triangleq \sigma_i^2 \tag{32}$$

where

$$A_{CL} \triangleq A - BG(I+EG)^{-1}F \tag{33a}$$

$$C_{CL} \triangleq \begin{bmatrix} C \\ -G(I+EG)^{-1}F \end{bmatrix} \tag{33b}$$

$$\epsilon_i \triangleq \sigma_i, \qquad i = 1,\dots,n_y \tag{33c}$$

$$\epsilon_{n_y+k} \triangleq \mu_k, \qquad k = 1,\dots,n_u \tag{33d}$$

The necessary conditions are obtained by minimizing the unconstrained function

$$\begin{aligned} V_{OIL} = {} & \mathrm{tr}X(Q + M^*G^*RGM) + \mathrm{tr}K(XA^*_{CL} \\ & + A_{CL}X + DWD^* + X_0) + \mathrm{tr}\Lambda_y(CXC^* - \sigma^2) \\ & + \mathrm{tr}\Lambda_u(GMXM^*G^* - \mu^2) \end{aligned} \tag{34}$$

where K is matrix of Lagrange-type parameters associated with equality constraint of Eq. (31), and the Kuhn-Tucker parameters associated with inequality constraints of Eq. (32) are Λ_y and Λ_u. The matrices Λ_y, Λ_u, σ, and μ are diagonal, with elements μ_i, σ_i, and

$$\Lambda_u = \mathrm{diag}[\Lambda_k, \Lambda_d, \Lambda_m]$$

where

$$\Lambda_k = \mathrm{diag}[\cdots \lambda_{k_i} \cdots]$$

$$\Lambda_d = \mathrm{diag}[\cdots \lambda_{d_i} \cdots]$$

$$\Lambda_m = \mathrm{diag}[\cdots \lambda_{m_i} \cdots]$$

The necessary conditions are obtained by extremizing $V = V_{OIL}$, to wit,

$$\frac{\partial V}{\partial K} = 0 = XA^*_{CL} + A_{CL}X + DWD^* + X_0 \tag{35a}$$

$$\begin{aligned} \frac{\partial V}{\partial X} = 0 = {} & KA_{CL} + A^*_{CL}K + Q + M^*G^*(R + \Lambda_u)GM \\ & + C^*\Lambda_yC \end{aligned} \tag{35b}$$

$$\frac{1}{2}\frac{\partial V}{\partial G_{ii}} = 0 = \left[(R+\Lambda_u)GMXM^*\right]_{ii} - \left[(I+EG)^{-1}FXKB\right]_{ii} \tag{35c}$$

$$-\left[(R+\Lambda_u)G(I+EG)^{-1}EMXM^*G\right]_{ii}$$

$$-\left[(I+EG)^{-1}FXKBG(I+EG)^{-1}E\right]_{ii} \tag{35d}$$

$$\Lambda_{y_{ii}}\left[(CXC^*)_{ii}-\sigma_i^2\right]=0 \tag{35e}$$

$$\Lambda_{u_{ii}}\left[(GMXM^*G^*)_{ii}-\mu_i^2\right]=0 \tag{35f}$$

where Eqs. (35e) and (35f) are necessary conditions imposed on the Kuhn-Tucker parameters.

The previous OIL_2 problem penalizes and constrains $u(t)$, the "control signal" resulting from the passive changes in the structure due to ΔM, ΔD, and ΔK. It might be more natural to penalize and constrain the *physical* elements in ΔM, ΔD, and ΔK. This can be accomplished by modifying the problem as follows.

$$\hat{B}^* \triangleq \begin{bmatrix} B_k^* & 0 \\ 0 & B_d^* \\ B_m^*(M+\Delta M)^{-1}(K+\Delta K) & B_m^*(M+\Delta M)^{-1}(D+G+\Delta D) \end{bmatrix}$$

L_2 Output/Parameter Constrained (OPL_2) Problem

Given the system of Eq. (11), find the diagonal G to minimize

$$\sum_{i=1}^{nx+nw}\int_0^\infty x^*(i,t)Qx(i,t)\,dt + \mathrm{tr}RG^2$$

subject to inequality constraints

$$\left[\sum_{i=1}^{nx+nw}\int_0^\infty y_\alpha^2(i,t)\,dt\right]^{1/2} \le \sigma_\alpha, \qquad \alpha=1,\ldots,n_y$$

$$[G_{ii}^2]^{1/2} \le \mu_k, \qquad k=1,\ldots,n_u$$

Problem OPL_2 leads to the augmented cost function

$$V_{OPL} = \mathrm{tr}XQ + \mathrm{tr}RG^2 + \mathrm{tr}K(XA_{CL}^* + A_{CL}X + DWD^*$$

$$+X_0) + \mathrm{tr}\Lambda_y(CXC^*-\sigma^2) + \mathrm{tr}\Lambda_u(G^2-\mu^2) \tag{36}$$

and the necessary conditions

$$\frac{\partial V_{OPL}}{\partial X} = 0 = KA_{CL}^* + A_{CL}K + C^*\Lambda_yC + Q \tag{37}$$

$$\frac{\partial V_{OPL}}{\partial K} = 0 = XA_{CL}^* + A_{CL}X + DWD^* + X_0 \tag{38}$$

$$\frac{1}{2}\frac{\partial V_{OPL}}{\partial G_{k_{ii}}} = 0 = (R_{k_{ii}}+\lambda_{k_i})G_{k_{ii}}$$

$$-\left\{B_k^*[XK]_{12}(M+\Delta M)^{-1}B_k\right\}_{ii} \tag{39}$$

$$\frac{1}{2}\frac{\partial V_{OPL}}{\partial G_{d_{ii}}} = 0 = (R_{d_{ii}}+\lambda_{d_i})G_{d_{ii}}$$

$$-\left\{B_d^*[XK]_{22}(M+\Delta M)^{-1}B_d\right\}_{ii} \tag{40}$$

$$\frac{1}{2}\frac{\partial V_{OPL}}{\partial G_{m_{ii}}} = 0 = (R_{m_{ii}}+\lambda_{m_i})G_{m_{ii}}$$

$$-\Big\{B_m^*(M+\Delta M)^{-1}\left[(K+\Delta K)[XK]_{12}+(D+\Delta D\right.$$

$$\left.+G)[XK]_{22}(M+\Delta M)^{-1}B_m\right\}_{ii} \tag{41}$$

$$0=\Lambda_{y_{ii}}\left[(CXC^*)_{ii}-\sigma_i^2\right] \tag{42}$$

$$0=\lambda_{k_i}\left[G_{k_i}^2-\mu_{k_i}^2\right] \tag{43}$$

$$0=\lambda_{d_i}\left[G_{d_{ii}}^2-\mu_{d_i}^2\right] \tag{44}$$

$$0=\lambda_{m_i}\left[G_{m_{ii}}^2-\mu_{m_i}^2\right] \tag{45}$$

Equations (39-41) can be solved for G_k, G_d, and G_m to give the compact form for G

$$G_{ii} = -\left[\hat{R}^{-1}\hat{B}XKI(M+\Delta M)^{-1}B\right]_{ii}, \qquad i=1\rightarrow n_u \tag{46}$$

where

$$\hat{R}\triangleq R+\Lambda_u, \qquad I^*=[0 \quad I]$$

$$B \triangleq [B_k \quad B_b \quad B_m]$$

These conditions suggest the following computational algorithm.

Algorithm OPL_2

Step 1) Input data.

$$\{M,D,K,G,B_k,B_d,B_m,\mu,\sigma,\Lambda_u(0)=\mu^{-2},\Lambda_y(0)=\sigma^{-2},X_0,W,\epsilon,R,Q,G(0)\}$$

Step 2) Solve Eq. (37) for K.
Step 3) Solve Eq. (38) for X.
Step 4) Solve Eq. (46) for G.
Step 5) Update Λ_u,Λ_y by

$$\Lambda_y(k+1)=\Lambda_y(k)\mathrm{diag}\left[\cdots\frac{(CXC^*)_{ii}}{\sigma_i^2}\cdots\right]^\alpha \tag{47}$$

$$\Lambda_u(k+1)=\Lambda_u(k)\mathrm{diag}\left[\cdots\frac{G_{ii}^2}{\mu_i^2}\cdots\right]^\alpha \tag{48}$$

[adjust α and β for best convergence (problem dependent)].

Step 6) Return to step 2 unless

$$\|\Lambda_y\,\mathrm{diag}[\cdots(CXC^*)_{ii}-\sigma_i^2\cdots]\|<\epsilon_2$$

$$\|\Lambda_u\,\mathrm{diag}[\cdots G_{ii}^2-\mu_i^2\cdots]\|<\epsilon_2 \tag{49}$$

V. Examples

The configuration in Fig. 1 was used in four examples with $Q=0$, $\mu=\infty$, $R=I$, and $DWD^*+X_0=I$. The first two examples are the OIL_2 problem with and without change in the mass elements, and the last two are the OPL_2 problem with and without the change in the mass elements. The data are defined as follows.

$\lambda_i \triangleq$ optimal ith eigenvalue pair
$\zeta_i \triangleq$ optimal ith damping ratio (initial damping is zero)
$y_i \triangleq \|y_i(\cdot)\|_\infty^2 \triangleq$ output L_∞ norm squared
Δm_i, Δd_i, $\Delta k_i \triangleq$ change in the physical elements (masses, dampers, and springs)
$V_u \triangleq \sum_{i=1}^{nx+nw}\int_0^\infty u^*(i,t)Ru(i,t)\,dt$, minimum input signal power

$V_G \triangleq \text{trace}(RG^2)$; minimum weighted norm on the physical elements $[\Delta m_i, \Delta d_i, \Delta k_i]$

Q = output weighting matrix = 0

R = input weighting max = I

The initial mass, spring, and damper values for all examples are

$$[\cdots m_i \cdots] = [4 \quad 2 \quad 10]$$

$$[\cdots k_i \cdots] = [2 \quad 2 \quad 4 \quad 1 \quad 1 \quad 3]$$

$$[\cdots d_i \cdots] = [0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0]$$

and the output L_∞ norm constraints are

$$[\cdots \epsilon_i \cdots] = [1.4174 \quad 1.0690 \quad 3.7166 \quad 0.8470 \quad 0.8456 \quad 1.6644]$$

The results are displayed in Table 1.

In the OIL_2 examples (cases 1 and 2 in Table 1), the damping was increased significantly. Comparing cases 1 and 2, we note that allowing the mass to change reduced the control power V_u by more than 20%. Note also that all three masses were reduced in the case 1, giving a total mass reduction of 19%. Only one of the six output constraints is not binding in both examples. This output is y_3, the displacement of mass 3.

In the OPL_2 examples, the damping factor also increased for every eigenvalue. These two examples (cases 3 and 4) also show that V_G decreased when the mass was allowed to change. If the masses are part of the set of variables, then the sum of the norms on all of the elements in these examples reduced by 31%. Also note that the masses were reduced in example 3 even more than in example 1, indicating that the direct minimization of structural parameters changes may yield solutions with smaller mass than the indirect approach of minimizing signal power (as typically done in control designs). In both of these examples, all output constraints were binding, indicating an efficiency in the design (the structure has been changed as much as the response constraints will allow).

VI. Large-Dimension Problems and Practical Applications

The spring-mass-damper examples offered earlier are small and have simple connectivity. The particular diagonal gain formulation shown is directly implementable on structural/mechanical systems having two-dimensional connectivity elements, i.e., trusses and spring-mass systems. The number of control variables equals the number of structural members and can become large for large trusses. Solutions of the previous sample problems required on the average of five iterations and less than 2 min on a 25MHz desktop computer while computing, displaying on video, and storing a comprehensive set of information for diagnostics that would be nonessential in routine operation. The use of a streamlined version on supercomputers should allow a considerable increase in degrees of freedom, i.e., structural members. A benchmark verification of the algorithm efficiency on large systems has not been conducted at this time for two reasons. First, it is believed that the algorithm, as is, is useful for many structural/mechanical design problems. Second, the algorithm's performance is highly dependent on the selection of weighting matrices and initial designs. Current work is directed at clarifying these selections.[19]

More complex structures such as beams and plates can be formulated similarly. In the basic finite element connectivity formulation for such structures, the gain matrices of Sec. II have block diagonal form in which each finite element represents one block in the master gain matrix. Each element can then be converted to diagonal form if so desired. However, used directly, the previous algorithms will require using elements with nonisotropic properties and have not been explored.

Optimal location of structural members, i.e., selection of the B matrix, is a problem similar to that of output feedback control in active systems and any method applicable there is a candidate. (Structural designers frequently use strain energy as a criterion.) Two differences have some impact on the process. First, because of their relatively low cost and lack of power need, the number of passive members that may be considered as controls is less restrictive and, second, the *initial* design (i.e., the connectivity) is usually generated by geometry and loads considerations. Dynamics design then becomes a modification problem.

It should be noted that negative closed-loop gains have no meaning in passive structures and must be avoided in the design. The algorithms were programmed with limits to prevent this. However, it was discovered that the algorithms converged most rapidly by choosing an initial condition that increased

Table 1 Numerical results

Case	$\frac{\Delta m_i}{m_i}$, %	Δd_i	$\frac{\Delta k_i}{k_i}$, %	$\frac{\Vert y_i\Vert_\infty^2}{\epsilon_i^2}$	λ_i	ζ_i	$Vu(VG)$
1, OIL_2	−28	2.54	23	1	$-1.88 \pm j0.21$	0.99	39.86
	−27	2.86	33	1	$-0.81 \pm j0.83$	0.70	
	−13	2.40	−2	0.84	$-0.22 \pm j0.74$	0.29	
		1.10	−17	1			
		1.05	−49	1			
		1.34	−43	1			
2, OIL_2	0	3.57	67	1	$-1.62 \pm j0.75$	0.91	49.98
	0	3.35	61	1	$-0.74 \pm j0.84$	0.66	
	0	2.68	1	0.92	$-0.22 \pm j0.70$	0.29	
		1.41	24	1			
		1.16	−43	1			
		1.41	−43	1			
3, OPL_2	−43	1.87	15	1	$-3.17 + j0$		(26.47)
	−54	1.79	−20	1	$-1.28 + j0$	1.1	
	−10	2.60	−6	1	$-0.81 \pm j0.80$	0.72	
		1.03	−37	1	$-0.22 \pm j0.69$	0.30	
		0.95	−90	1			
		1.17	−61	1			
4, OPL_2	0	3.38	72	1	$-1.38 \pm j0.84$	0.85	(38.13)
	0	2.44	37	1	$-0.67 \pm j0.82$	0.63	
	0	2.89	6	1	$-0.21 \pm j0.68$	0.30	
		1.55	29	1			
		1.09	−71	1			
		1.05	−70	1			

both stiffness and damping significantly and by allowing the design to reduce these. The limit feature was never activated.

VII. Conclusions

This paper develops an iterative algorithm to redesign lumped parameter structures to guarantee hard (absolute) amplitude constraints on the dynamic response in the presence of an arbitrary disturbance with a specified energy level. The redesign allows changes in mass, stiffness, and damping. Examples demonstrate rapid convergence, although convergence is not proved. The examples show a significant reduction in mass while improving the dynamic response to satisfy output constraints in the presence of any disturbance within a specified energy level. This is because it takes less energy to move smaller masses, but the particular response constraints also influence the results. Future work will investigate more realistic structures to include parameters appearing nonlinearly in the mass, stiffness, and damping matrices.

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